Interaction Energy in Geometrostatics

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The "geometrodynamical" description of particles by means of topological features of empty space-time is applied here to the case of N charged masses which are momentarily at rest. These particles are represented by Einstein-Rosen bridges in a mainfold which satisfies the time-symmetric initial value equations of gravitation and electromagnetism. Invariant definitions are given for the total mass energy of the system, and for the "bare mass" of each Einstein-Rosen bridge. These masses characterize various asymptotically Schwarzschildian regions of the manifold and are, therefore, conserved in time. The total mass of the system differs from the sum of the bare masses by contributions from the gravitational and electrostatic interaction energies. It is shown that the interaction energy is always negative, and that it reduces to the classical expression in the limit of large separation between the masses. The shape of the minimal surface associated with each Einstein-Rosen bridge, another invariant feature of the "particle," is discussed. The minimal surfaces are also used to characterize manifolds which can be interpreted as a closed universe containing $N+1$ "particles."

I. INTRODUCTION

THE Einstein-Maxwell equations of gravitation and electromagnetism in source-free space

$$R_\mu{}^\nu-\tfrac{1}{2}\delta_\mu{}^\nu R=F_{\mu\alpha}F^{\alpha\nu}-\tfrac{1}{4}\delta_\mu{}^\nu F_{\alpha\beta}F^{\alpha\beta},\qquad F^{\mu\nu}{}_{;\mu}=0,\quad F_{[\mu\nu,\kappa]}=0,\tag{1}$$

admit many nontrivial solutions of physical interest if the underlying manifold is permitted to have a sufficiently general topology. In particular, these equations have been discussed in multiply connected manifolds, and it has been shown that certain solutions imitate the behavior of real masses and charges.[1] In the following we discuss further properties of such solutions which can be interpreted in terms of particles and their interactions. (The term "particle" is used throughout this paper as a synonym for "geometrodynamical entity." It should be understood that these constructs of *classical* geometrodynamics have no direct connection whatever with the particles of the real physical world.)

The well-known solutions of Schwarzschild and Reissner-Nordström for a spherically symmetric geometry endowed with mass and charge provide examples of nontrivial solutions on multiply connected manifolds. The maximum analytic continuation of these solutions,[2] viewed at a particular instant of time, has a topology and curvature as indicated in Fig. 1. The tube connecting the two asymptotically flat sheets will be called an Einstein-Rosen bridge[3]; it is a particular example of a model for mass and charge in the realm of source-free gravitation and electromagnetism.

In order to generalize this type of solution to a geometry containing several bridges (a model of several masses), one may follow a stepwise procedure: (a) On some space-like surface Σ solve those equations in (1) which have $\nu=0$. These involve only the "initial data": the electromagnetic field tensor on Σ, the induced metric on Σ, and its first normal derivative. (b) Then use the remaining equations to find the time development—the initial value equations will automatically be satisfied for all later times. For the purpose of our computation it is sufficient to carry out step (a), which is simpler than solving the full set of equations.

No nontrivial genuinely static solutions of Einstein's equations exist; the next simplest are the "time-symmetric," or "momentarily static" solutions. We confine attention to this case.[4] In step (a) only two equations

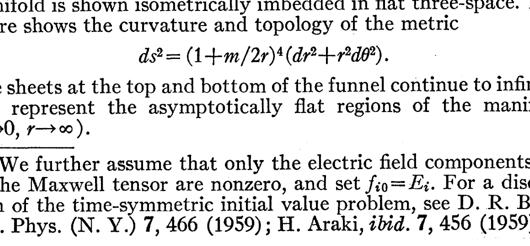

FIG. 1. A two-dimensional analog of the Schwarzschild-Kruskal manifold is shown isometrically imbedded in flat three-space. The figure shows the curvature and topology of the metric

$$ds^2=(1+m/2r)^4(dr^2+r^2d\theta^2).$$

The sheets at the top and bottom of the funnel continue to infinity and represent the asymptotically flat regions of the manifold ($r\to 0$, $r\to\infty$).

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[1] C. W. Misner and J. A. Wheeler, Ann. Phys. (N.Y.) **2**, 525 (1957). This and other important papers are collected in J. A. Wheeler, *Geometrodyanmics* (Academic Press Inc., New York, 1962).

[2] M. D. Kruskal, Phys. Rev. **119**, 1743 (1960); J. C. Graves and D. R. Brill, *ibid.* **120**, 1507 (1960).

[3] A. Einstein and N. Rosen, Phys. Rev. **48**, 73 (1935).

[4] We further assume that only the electric field components f_{i0} of the Maxwell tensor are nonzero, and set $f_{i0}=E_i$. For a discussion of the time-symmetric initial value problem, see D. R. Brill, Am. Phys. (N. Y.) **7**, 466 (1959); H. Araki, *ibid.* **7**, 456 (1959).

remain to be solved[5]:

$$R=2E_iE^i,\qquad E^i{}_{;i}=0. \tag{2}$$

The coordinates used to describe our manifold are chosen in analogy to the isotropic coordinates of the Schwarzschild and Reissner-Nordström solution: The single coordinate "patch" covering the entire manifold consists of Euclidean space with as many points removed as there are Einstein-Rosen bridges. The criteria which must be checked for an acceptable solution are (a) regularity and (b) completeness in the remaining manifold. By way of illustration, consider again the case of the Schwarzschild solution. Here the coordinate "patch" is Euclidean space with the origin $r=0$ removed. The coefficients of the isotropic metric on Σ are finite in this coordinate patch, and a study of the solution in the neighborhood of $r=0$ reveals the geometry of Fig. 1, which is indeed complete.[6]

II. INITIAL GEOMETRY AND FIELD FOR N CHARGED BRIDGES

We consider here only a metric and electric field of the simple form

$$ds^2=(\chi\psi)^2 ds_F^2, \tag{3a}$$

with ds_F^2 the metric field for flat space on Σ, and

$$E_i=-\phi_{,i}=[\ln(\chi/\psi)]_{,i}. \tag{3b}$$

(This form of the metric and electric field is a natural generalization of the Reissner-Nordström solution for a *single* charged Einstein-Rosen bridge.) The time-symmetric initial values equations (2) then take on the simple form

$$\Delta\chi=0,\quad \Delta\psi=0, \tag{4}$$

Δ being the flat-space Laplacian operator. These equations must be supplemented by appropriate boundary conditions. We use here the conditions of regularity and asymptotic flatness:

$$\begin{aligned}&\text{(i)}\quad \chi>0,\quad \psi>0\quad \text{everywhere on } \Sigma,\\ &\text{(ii)}\quad \chi,\psi\rightarrow 1\quad \text{as}\quad r\rightarrow\infty.\end{aligned} \tag{5}$$

Let r_i be the Euclidean distance in Σ from a field point to the ith deleted point; then the general solution of (4) and (5) is given by

$$\chi=1+\sum_{i=1}^{N}\alpha_i/r_i,\quad \psi=1+\sum_{i=1}^{N}\beta_i/r_i, \tag{6}$$

where $\alpha_i>0$, $\beta_i>0$ for all i in accord with condition (i). The same condition excludes higher multipole terms in (6).

Completeness

Since our solution is asymptotically flat at $r_i\rightarrow\infty$, the only regions to be investigated for completeness are the neighborhoods of a deleted point. We, therefore, discuss the metric in the limit,

$$r_i\rightarrow 0,\quad r_j\rightarrow r_{ij}\quad (j\neq i) \tag{7}$$

(for a particular value of i). Here r_{ij} denotes the Euclidean distance on Σ between the ith and jth deleted point. In this limit the line element takes the form

$$ds^2\rightarrow[(\alpha_i^2\beta_i^2/r_i^4)(1+A_ir_i/\alpha_i)^2(1+B_ir_i/\beta_i)^2+O(r_i^2)][dr_i^2+r_i^2d\Omega^2], \tag{8}$$

where we have set

$$A_i=1+\sum_{j\neq i}\frac{\alpha_j}{r_{ij}}\quad B_i=1+\sum_{j\neq i}\frac{\beta_j}{r_{ij}}, \tag{9}$$

and

$$d\Omega^2=d\theta^2+\sin^2\theta d\varphi^2.$$

Introduce a new coordinate

$$r_i'=\alpha_i\beta_i/r_i. \tag{10}$$

Then the line element takes the form, in the limit $r_i'\rightarrow\infty$,

$$ds^2\rightarrow[(1+A_i\beta_i/r_i')^2(1+B_i\alpha_i/r_i')^2+O(1/r_i'^2)]\times(dr_i'^2+r_i'^2d\Omega^2). \tag{11}$$

This expression shows that in this limit the space is also asymptotically flat. It is, therefore, complete. Thus, our metric [(3a) and (6)] describes a space with $N+1$ asymptotically flat regions, a generalization of the Reissner-Nordström solution, for which $N=1$ (Fig. 1). A two-dimensional analog of the case $N=2$, with three asymptotically flat sheets, is shown in Fig. 2. Note that the two lower sheets are separate, and it is not possible to pass from one to the other except by way of the upper sheet. (Similar solutions in which the lower sheets can be identified to form a single sheet have been given by Misner[6] for the uncharged, and by Lindquist[7] for the charged case.)

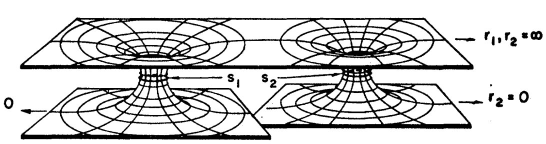

FIG. 2. A two-dimensional analog of the hypersurface of time symmetry of a manifold containing two "throats" is shown isometrically imbedded in flat three-space. The figure illustrates the curvature and topology for a system of two "particles" of equal mass m, and separation large compared to m, described by the metric

$$ds^2=(1+m/2r_1+m/2r_2)^4ds_F^2.$$

[5] A. Lichnerowicz, *Théories relativistes de la gravitation et de l'électromagnétisme* (Masson et Cie, Paris, 1955); Y. Fourès-Bruhat, J. Rat. Mech. Anal. **5**, 951 (1956). See also reference 1.

[6] See, for example, C. W. Misner, Ann. Phys. (N. Y.) (to be published).

[7] R. W. Lindquist, J. Math. Phys. (to be published).

III. PARTICLE INTERPRETATION

Bare Mass and Total Mass

Equation (11) shows that the $(N+1)$-sheeted solution asymptotically takes on a Schwarzschildian form on *each* sheet; therefore, a unique mass energy is associated with each asymptotic region, and can be computed by simple comparison with the terms in $1/r$ of the Schwarzschild solution:

$$ds^2_{\text{Schwarzschild}} \rightarrow [1+m/r+O(1/r^2)]^2(dr^2+r^2d\Omega^2). \quad (12)$$

Thus, the mass as measured in the ith sheet is given by

$$m_i = A_i\beta_i + B_i\alpha_i = \alpha_i + \beta_i + \sum_{j\neq i}(\beta_i\alpha_j + \beta_j\alpha_i)/r_{ij}. \quad (13)$$

Also, the mass on the $(N+1)$st ("upper") sheet, $r_i \rightarrow \infty$ (all i) is found directly from Eq. (6):

$$m_{N+1} = \sum_{i=1}^{N}(\alpha_i+\beta_i). \quad (14)$$

Although no particular sheet of our solution is geometrically preferred, the (unprimed) coordinates we are using are appropriate for an observer located on the $(N+1)$st sheet (corresponding results will hold for an observer on any of the other sheets). Such an observer would consider the system as composed of N "particles" which together produce a total mass energy $M = m_{N+1}$. The well-defined mass m_i associated with each particle then corresponds to the "bare mass" of the particle.

Charge

In addition to the mass, a measure of the flux of the electric field is also uniquely defined on each of the $N+1$ sheets. In accordance with the spirit of geometrodynamics, we associate a charge q_i with each Einstein-Rosen bridge, defined such that the flux on the ith sheet is given by $4\pi q_i$. We evaluate the flux through a large sphere, $r_i' \rightarrow \infty$, or equivalently, $r_i \rightarrow 0$:

$$q_i = (1/4\pi)\int E_i n^i dS. \quad (15)$$

Here E_i is given by (3b); n^i is the unit outward normal to the sphere $r_i' = \text{constant}$ and, hence, points inward in the r coordinates:

$$n^r = -1/\chi\psi,$$
$$n^\theta = n^\varphi = 0,$$

while $dS = (\chi\psi)^2 r^2 \sin\theta d\theta d\varphi$ is the element of area on the sphere. Therefore,

$$\begin{aligned} q_i &= -(1/4\pi)\int (\psi\partial\chi/\partial r - \chi\partial\psi/\partial r)r^2 d\Omega \\ &= A_i\beta_i - B_i\alpha_i \\ &= \beta_i - \alpha_i + \sum_{j\neq i}(\beta_i\alpha_j - \beta_j\alpha_i)/r_{ij}. \end{aligned} \quad (16)$$

Due to conservation of flux, the charge on the $(N+1)$st sheet is, of course, simply the sum of the individual charges q_i.

By comparing (13) and (16), and recalling that α_i and β_i are positive, one readily sees that the magnitude of q_i can never exceed m_i. This restriction on the charge-to-mass ratio in general relatively—already present in the Reissner-Nordström solution—is a quite general consequence of the assumption of regularity, and is insensitive to the particular model used to represent the particles.

The constants of integration α_i and β_i are determined by the physical quantities m_i and q_i via Eqs. (13) and (16). We give the inverse formulas to order $1/r_{ij}$:

$$\begin{aligned} \alpha_i &\approx \tfrac{1}{2}(m_i - q_i)[1 - \tfrac{1}{2}\sum_{j\neq i}(m_j+q_j)/r_{ij}], \\ \beta_i &\approx \tfrac{1}{2}(m_i + q_i)[1 - \tfrac{1}{2}\sum_{j\neq i}(m_j-q_j)/r_{ij}]. \end{aligned} \quad (17)$$

Interaction Energy

If there were no interaction between the particles, the total mass would equal the sum of the bare masses. However, one sees from (13) and (14) that

$$M = \sum_{i=1}^{N} m_i - \sum_{i=1}^{N}\sum_{j\neq i}(\alpha_i\beta_j + \alpha_j\beta_i)/r_{ij}. \quad (18)$$

The difference between M and $\sum m_i$ is to be attributed, as in other field theories, to the energy of the electrostatic and gravitational interaction between the N masses. Equation (18) shows that the interaction mass energy

$$m_{\text{int}} = M - \sum_{i=1}^{N} m_i = -\sum_{i=1}^{N}\sum_{j\neq i}(\alpha_i\beta_j + \alpha_j\beta_i)/r_{ij} \quad (19)$$

is always *negative*, corresponding to attractive forces between the masses. This is reasonable, since for a nonsingular solution the "charge" associated with each "particle" cannot exceed its mass, as shown above.

Consider now the limit when the masses are far apart, so that Eq. (17) applies. The expression (19) for the interaction energy then takes the approximate form

$$m_{\text{int}} \approx -\sum_{i=1}^{N}\sum_{j\neq i}(m_i m_j - q_i q_j)/r_{ij}. \quad (20)$$

This is precisely the expression for the energy of interaction according to Newtonian gravitation and flat-space Maxwell theory, as one would expect in this limit.

IV. MINIMAL SURFACES

In the previous sections we have singled out one sheet of the initial hypersurface to represent the "rest of the universe." Since the metric is everywhere conformally flat, and asymptotically flat on each sheet, we

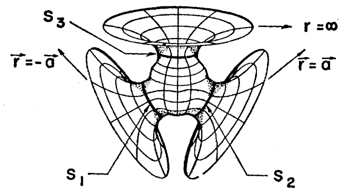

FIG. 3. The manifold illustrated here is of the same type as that of Fig. 2, but the separation is comparable to the masses: $\alpha_i=\beta_i=r_{12}(i=1,\,2)$. The metric can be written in the form

$$ds^2=(1+2a/|\mathbf{r}-\mathbf{a}|+2a/|\mathbf{r}+\mathbf{a}|)^4(dr^2+r^2d\Omega^2).$$

Here the particles are so close together that a third minimal surface, S_3 appears in addition to the two surfaces S_1 and S_2 already present in Fig. 2. The three sheets are, in fact, completely equivalent, for the inversion maps $\mathbf{r}\pm\mathbf{a}\rightarrow(2a/|\mathbf{r}\pm\mathbf{a}|)^2(\mathbf{r}\pm\mathbf{a})$ are isometries. Viewed from the top sheet the manifold represents a composite particle; viewed from a symmetrical position in the center of the figure it represents three "particles" symmetrically placed in a closed universe.

could equally well have chosen any other, say, the ith, for this purpose. This would also provide a metric and electric field of the form of Eq. (3), and all the preceding formulas would apply to this new interpretation. However, the total mass (now defined as the mass associated with the ith sheet) and interaction energies would have different numerical values. Note that this total mass satisfies the polygon inequality

$$m_i<\sum_{j=1\,(j\neq i)}^{N+1} m_j. \tag{21}$$

Yet in some cases the geometry itself singles out one sheet above all others. For example, the three sheets of Fig. 2 are joined together along two surfaces S_1, S_2 of minimal area. Each minimal surface can be deformed continuously into a sphere lying in the asymptotic region of one of the *lower* sheets. There exists no such deformation of either minimal surface into an asymptotic sphere in the upper sheet. Similarly, the $(N+1)$-sheeted manifold admits at least N-independent minimal surfaces, each of which can by a continuous deformation be associated with one of these sheets.[8] If there are precisely N minimal surfaces, it is natural to interpret the corresponding N sheets as "particles," and the remaining $(N+1)$st sheet as the "arena" in which these particles move about. This happens whenever the N particles are well separated (m_i and $m_j\ll r_{ij}$).

When the "particles" are very close together it is found that another minimal surface appears. This additional surface is associated with the remaining $(N+1)$st sheet, and, thus, leads to an interpretation of the manifold Σ as a model of a single composite particle with an internal structure, immersed in an asymptotically flat space (Fig. 3). An alternative interpretation arises if one treats the sheets symmetrically and focuses attention on the region of concentration of curvature. This region has the topology of a 3-sphere with $N+1$ Einstein-Rosen bridges attached. It can be viewed, therefore, as a model of a "closed universe" containing $N+1$ particles.[9]

Approximate Shape of Minimal Surfaces

To illustrate these features we examine the shape of the minimal surfaces in the two limiting cases of very small and very large ratio of mass-to-separation distance. Let (r,θ,φ) be a set of spherical coordinates based at the point $r_i=0$, and let the equation for the minimal surfaces S_i associated with the ith particle be expressed as

$$r=r(\theta,\varphi).$$

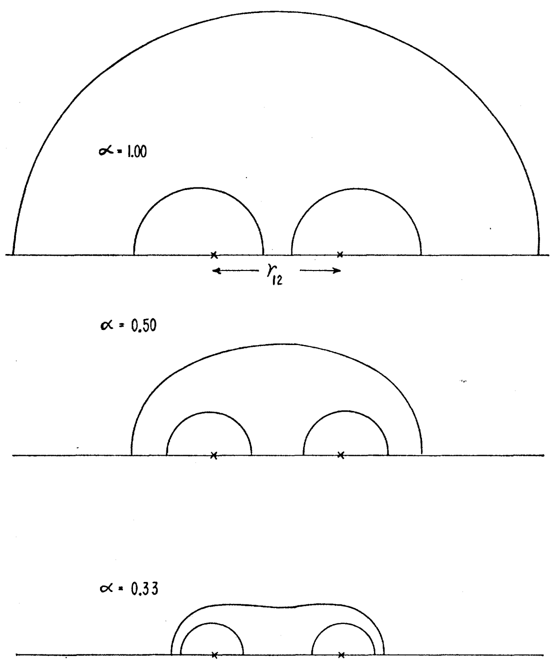

FIG. 4. Results of a numerical determination of the minimal surfaces for two equal uncharged Einstein-Rosen bridges are shown for several different values of the ratio (mass/separation). This ratio is measured by the parameter $\alpha=\alpha_1/r_{12}(=\alpha_2/r_{12}=\beta_1/r_{12}=\beta_2/r_{12}$, since $m_1=m_2$ and $q_1=q_2=0)$. The radius $r(\theta)$ of the minimal surface was expanded in a truncated series of Legendre polynomials:

$$r(\theta)=\sum_{l=0}^{L} c_l P_l(\cos\theta),$$

and the coefficients c_l chosen to minimize the integral (24). The figure illustrates the shape of a cross section of each minimal surface as seen in the flat imbedding space. As α increases, the surface enclosing each particle becomes more and more highly distorted while the outer surface approaches the sphere $r=2r_{12}$. Conversely, as α decreases the inner surfaces approach spheres of radii $\alpha r_{12}/(1+\alpha)$, while the outer surface begins to pinch off; for $\alpha\leq 0.32$ no minimal surface enclosing both particles was found.

[8] The set of all 2-surfaces in Σ can be divided into homology classes, and each class (not ~ 0) contains at least one minimal surface. This surface will not necessarily be connected, but we can find at least N connected minimal surfaces which belong to distinct, linearly independent homology classes, since the second homology group of Σ has dimension N. Furthermore, each such minimal surface will be homologous to a sphere lying in the asymptotic region on one of the $N+1$ sheets.

[9] R. W. Lindquist and J. A. Wheeler, Rev. Mod. Phys. **29**, 432 (1957).

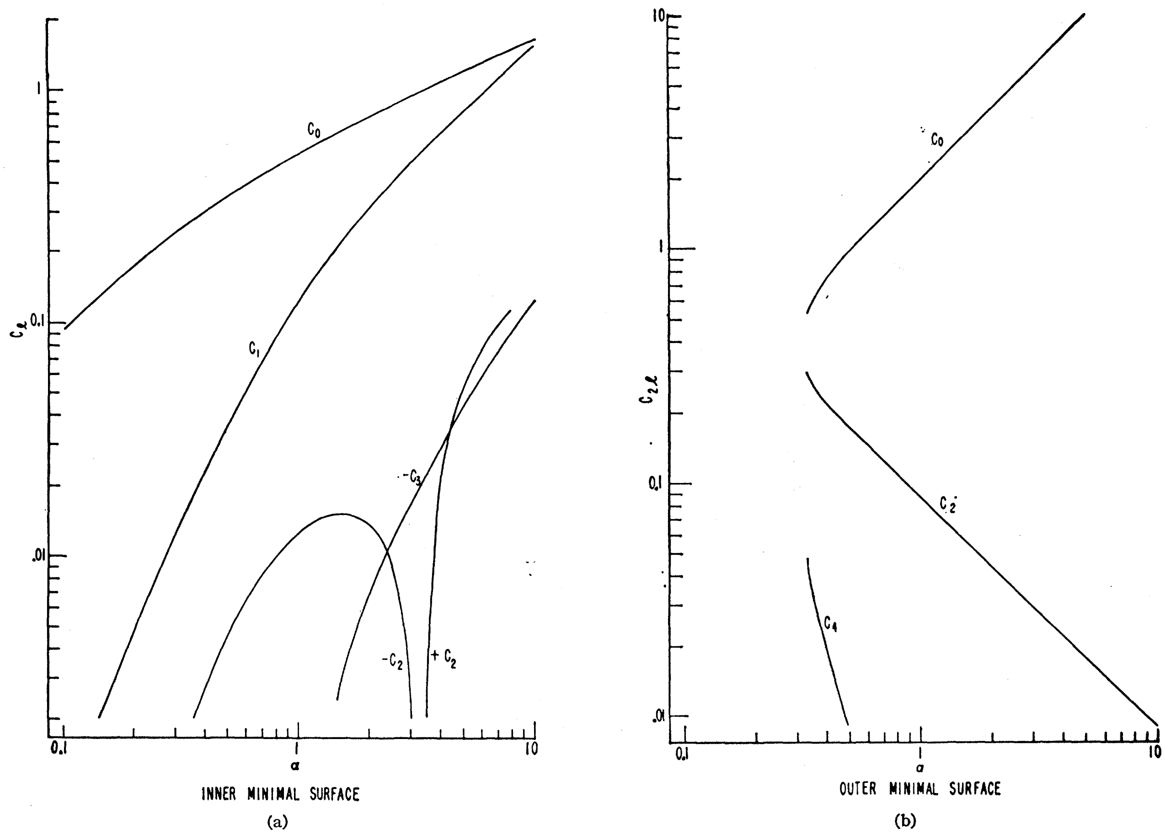

FIG. 5. Graphs of the coefficients c_l in the series expansion

$$r(\theta)=\sum_{l=0}^{L} c_l P_l(\cos\theta)$$

for the minimal surfaces of two equal uncharged Einstein-Rosen bridges [cf., Fig. 4]. In (a) the radius vector is drawn from the right-hand particle [i.e., $r(\theta)=r_2$], while in (b) it is drawn from the midpoint between them.

The initial value metric (3a) induces on S_i a two-dimensional submetric

$$d\sigma^2=[f(r(\theta,\varphi),\theta,\varphi)]^2[(r^2+(\partial r/\partial\theta)^2)d\theta^2 +(r^2\sin^2\theta+(\partial r/\partial\varphi)^2)d\varphi^2], \quad (22)$$

with

$$f=\chi\psi. \quad (23)$$

The condition that S_i be a minimal surface can be formulated by the variational principle

$$0=\delta\int_{S_i}({}^2g)^{1/2}d\theta d\varphi \equiv\delta\int\int f^2\left[r^2+\left(\frac{\partial r}{\partial\theta}\right)^2\right]^{1/2} \times\left[r^2\sin^2\theta+\left(\frac{\partial r}{\partial\varphi}\right)^2\right]^{1/2}d\theta d\varphi, \quad (24)$$

with $r(\theta,\varphi)$ allowed to vary freely, subject only to the condition that it describe a closed differentiable 2-surface enclosing the given particle.

When the particles are well separated the minimal surface can be approximated by a sphere:

$$r(\theta,\varphi)=r_0,$$

with

$$f\approx[A_i+(\alpha_i/r)][B_i+(\beta_i/r)], \quad (25)$$

[cf., Eq. (9)]. Then Eq. (24) implies

$$\partial(f^2r)/\partial r\,|_{r=r_0}=0, \quad (26)$$

and, consequently,

$$r_0^2\equiv\alpha_i\beta_i/(A_iB_i). \quad (27)$$

In the limit $r_{ij}\to\infty$ this reduces to the familiar expression for the "Schwarzschild radius" (expressed in isotropic coordinates):

$$r_S^2=\alpha_i\beta_i=\tfrac{1}{4}(m_i^2-q_i^2). \quad (28)$$

On the other hand, when the particles are very close together one again finds a minimal surface of approximately spherical shape, this time enclosing all of them.

Assuming that $r_0 \gg r_{ij}$ (for all i and j), one sets

$$f \approx [1+(\sum \alpha_i)/r][1+(\sum \beta_i/r)], \qquad (29)$$

and obtains from (26)

$$r_0^2 = (\sum \alpha_i)(\sum \beta_i) = \tfrac{1}{4}(m_{N+1}^2 - q_{N+1}^2). \qquad (30)$$

Hence, this additional minimal surface exists (and is nearly spherical) whenever the inequalities

$$m_{N+1} \gg r_{ij}, \quad m_{N+1} \gg |q_{N+1}|, \qquad (31)$$

are satisfied. The remaining N sheets also have their associated minimal surfaces; however, these are highly distorted, so that one must resort to numerical solution of the variational principle (24) to locate them. Figures 4 and 5 display the results of such a numerical investigation for the special case $N=2$ and $m_1=m_2$, $q_1=q_2=0$.

It can be shown that the minimal surfaces are elongated slightly along the lines joining them.[10] This type of distortion is just what one might expect by analogy with Newtonian tidal forces; indeed, the magnitude of the distortion is inversely proportional to the cube of the separation distance r_{ij} in the Newtonian limit, and increases as the ratio of mass to separation distance is increased. Thus, the deviation of each minimal surface from spherical symmetry provides another way to estimate the interaction between the N particles.

ACKNOWLEDGMENTS

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[10] J. A. Wheeler, Rev. Mod. Phys. **33**, 63 (1961).

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Solutions of the Density Matrix-Pairing Tensor Equations of Superconductivity Theory*

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An algorithm is provided for writing down explicitly all solutions of the density matrix-pairing tensor equations which arise in the generalized Bogoliubov-Valatin transformation approach to superconductivity theory. Certain simpler special cases are then examined. Finally reasons are given indicating that our solutions should provide a practical computational tool in many-body theory.

1. INTRODUCTION

A GENERALIZED Hartree-Fock method has been proposed by Bogoliubov[1] and Valatin[2] for investigating the quantum-mechanical problem posed by certain Hamiltonians of the form

$$H = \int dx dx' \psi^*(x) \epsilon(xx') \psi(x') + \frac{1}{2} \int dx dx' dx_1 dx_1' \times \psi^*(x) \psi^*(x') W(xx'x_1x_1') \psi(x_1') \psi(x_1). \qquad (1)$$

The letter x denotes the space and spin coordinates of a single particle, while ψ is the usual field operator which may be written

$$\psi = \xi^T a, \qquad (2)$$

where ξ is a column vector of orthonormal single-particle wave functions and a is a column vector of the corresponding destruction operators. We use the notation that ξ^T denotes the transpose of ξ.

In the approach of Bogoliubov one specifies a set ξ and introduces a new set of creation and destruction operators through the transformation

$$a = U\alpha + V\alpha^\dagger, \qquad (3)$$

where U and V denote matrices operating on column vectors of destruction and creation operators as indicated. The requirements that the a's and the α's both satisfy the usual fermion anticommutation rules lead to certain conditions on U and V which may be written simply in the present notation as

$$UU^{*T} + VV^{*T} = \mathrm{I}, \quad UV^T + VU^T = 0. \qquad (4)$$

The procedure is then to define a trial variational wave function $\Psi(U,V)$ through

$$\alpha\Psi = 0, \qquad (5)$$

and calculate the corresponding energy. The latter

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[1] N. N. Bogoliubov, Dokl. Akad. Nauk. S.S.S.R. **119**, 244 (1958); N. N. Bogoliubov and V. G. Soloviev, *ibid.* **124**, 1011 (1959) [translation: Soviet Phys.—Dokl. **3**, 292 (1958); **4**, 143 (1959)]. N. N. Bogoliubov, Usp. Fiz. Nauk. **67**, 549 (1959) [translation: Soviet Phys.—Usp. **2**, 236 (1959)].

[2] J. G. Valatin, Phys. Rev. **122**, 1012 (1961).